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A MIDDLEMAN WITH IMPERFECT INFORMATION

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by

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1. Introduction

A great deal of attention has recently been paid to microeconomic aspects of decision-making by economic agents under incomplete information. See [7~10] and [12]. Leijonhufvud, for example, considered a quantity adjustment model derived from the Marshallian homeostat [6]. In [6] the good is assumed perishable and the aspect of inventory holding has not been discussed. With no inventory (by assuming that the goods are nonstorable), decision-making processes of economic agents are sufficiently different from those in which inventories are held, and in a sense simpler.

In an economy of uncertain and imperfect information, however, ex ante plans of households and firms will not be generally coordinated. What they plan are not what they end up doing. In such an economy, one expects inventories to play an important role as buffer against the consequences of mismatched plans.1/ Inventory holding behavior of a store-keeper or a middleman in the world of uncertain and incomplete information has not been analyzed to any extent. See however [1,8].

In this paper, one model of a middleman who holds inventory is posited in Section 2 and his pricing behavior is examined under imperfect information assumptions. One thing this middleman knows with certainty is a fact that he is one of the many middlemen selling a homogeneous good or some close substitutes. He does not know exactly the current prices other middlemen are charging. This may be due to cost involved in gathering this piece of information and/or due to time involved in obtaining the price estimates which may make the estimates somewhat obsolete. We denote the middleman's price by p and some index of the "market" prices by $\hat{p}$. This middleman observes his sales i.e., how fast his inventory is being depleted weekly (monthly or quarterly) in response to the

price he sets and to exogenous disturbances which he cannot observe directly. In other words, in our model the price is the decision variable and the middleman's sales, i.e., the quantities sold, or equivalently the level of inventory and changes in the level of inventory constitute the only direct signals he receives, either to confirm or to refute some or all of his subjectively held beliefs about the market he is in and the demand conditions he faces.

The signal he observes mixes two kinds of messages, one being the feedback from his setting of price and the other originating from exogenous sources (random shifts of product demand, for example) and occuring independently of his decision. This complicates his decision process either to confirm or to refute some or all of his subjectively held beliefs about the market he is in and the demand condition he faces. He must, if he can, unscramble these two types of information since his appropriate response to the two kinds of events would be different. For example, an observed decline in his own sales over a succession of 'weeks' might mean either of two things. It could mean that competitors have lowered the prices they charge so that his own price is now out of line with prices prevailing elsewhere in the market; if this were the case, unless he lowers his price, he would be threatened by a continuing loss of customers to other sellers as consumers continue to learn about the prevailing price distribution. On the other hand, it could mean that market demand has declined generally with all sellers losing sales in roughly the same proportion -- a less alarming situation.

Our question, then, is whether he will be able to unscramble these two types of information in the signal he receives from the market, and respond properly to these two different types of information. In particular we examine if the middleman is able to "track" changes in the "price prevailing elsewhere" in the market. To examine a possible situation in which the question may be

answered affirmatively, we assume that amount, q, purchased by a customer arriving at his store is random and and is governed by a probability distribution function, which is taken to be a Gamma distribution $\Gamma(q;\alpha,\mu)$. The parameters, α and μ, which distinguish one Gamma distribution from the others are postulated to vary with the price differential, $p - \hat{p}$ causing relative shifts of demands and with the exogenous parameter Θ, indicating general shifts in demand. The precise nature of these dependence is described in Sec 2. The technique is independent of the specific distribution function, namely of the assumption of the Gamma distribution and is applicable to a wider class of distributions.

The price the middleman sets to maximize his long run profit rate depends crucially on two quantity signals and their price elasticities: on the average rate of sale per week, the deviation of his storage cost from the "average cost" to be specified later, and their subjectively conceived price elasticities. Another important factor is the customer arrival rate (which affects the rate of sales per week) and its price elasticities. The pricing behavior of the middleman turns out to be the same whether customer arrivals are modelled as a Poisson process or a uniform stream. We consequently perform the major part of our analysis of the implications of the posited dependent of the $\Gamma(q;\alpha,\mu)$ on $p - \hat{p}$ using a stream of customers arriving at a uniform rate. Point processes such as a Poisson process can also be used. They affect the analysis only through the average number of customers (which affects the average sales per week) and its price elasticity. We assume that the information on price differentials spreads only gradually among customers so that in the short run the rate of arrivals is not affected by the price differentials. We indicate in Sec. 5 how this assumption may be relaxed.

After establishing the method of discriminating between general and relative shifts, we examine auxiliary questions of how to detect parameter shifts and modify estimates of α,μ and the arrival rate parameter. Since these are statistical in nature and a body of literature exists on detection, we only briefly suggest a way these parameter values could be monitored.

2. Model

Consider a middleman who sells a storable consumption good, cans of soups, say.[2/] He faces a stream of customers, each of whom buys a random amount of the good. The distribution function which probabilistically governs demand by a customer is assumed to come from a class of distribution functions known to the middleman. The value of the parameter vector, which specifies a distribution function in the class completely, is unknown and depends on the price p the middleman sets and on prices of goods which are close substitutes in the market, as well as on some exogenous parameters, θ.

We shall refer to these prices loosely as prices prevailing elsewhere in the market. We use $\hat{p}$ to denote some index of these prices.

The exogenous parameter may represent for example, random shifts of demands for the good and its close substitutes. We may think of the middleman selling a slightly product differentiated good and trying to establish the right amount of price difference.[3/] A customer is indifferent then from whom he purchases the good, if the price is the same after allowance for whatever product differentiation is made.

Once the middleman sets a price p, he must decide either to maintain p or to change p in the fact of fluctuating demands he experiences over time.[4/] Fluctuations in the sales the middleman observes is partly due to the stochastic nature of demand for the good, partly due to exogenous random shifts, if any, of demands, and of course partly due to the fact that the

price differences $p-\hat{p}$ is not generally zero. The amount purchased by a customer, once he arrives at his store, responds without delay to the existing price differential $p-\hat{p}$, through the assumed dependence of the parameters of the distribution on it. Customer arrivals respond to price changes by gradual learning, hence do not change instantaneously. It is important to realize that the value of θ and the $p-\hat{p}$ differential are not directly observable. It is revealed indirectly through the quantity signals the middleman receives. Customers are assumed to arrive at a constant rate.[5/]

Once a customer arrives at a store, he purchases a random quantity $q \geq 0$ of the good. We assume q to be a continuous variable.[6/] Let $g(q;\theta,p,\hat{p})$ be its probability density function which is assumed to exist. (We drop θ, p and $\hat{p}$ from $g(q;\ldots)$.) One example of $g(\cdot)$ is the Γ-distribution. Its probability density function is

$$g(q) = \frac{\mu(\mu q)^{\alpha-1}e^{-\mu q}}{\Gamma(\alpha)} \tag{1}$$

where α and μ are the parameters of the distribution. α is known as the shape parameter, and μ is called the scale parameter. They depend on p, $\hat{p}$ and θ also. Their dependence on p, $\hat{p}$ and θ will be made explicit later. The mean is given by α/μ and the variance is α/μ^2. These parameter values are not known precisely to the middleman. We discuss a possible scheme for estimation and update of these parameters later in Section 3. We make an inessential assumption that α is a positive integer to facilitate analytical manipulations carried out in Appendix 1.

The middleman is assumed to be rational in that he sets p so as to maximize "average" profit over some long time interval.

The average profit rate π is aken to be given

$$\pi = R - C \tag{2}$$

where R is the average revenue rate from sales and where C is the cost (rate) composed of set-up costs and the storage cost. To focus our attention on the stochastic demand side, assume that he can get instant delivery of the supply of the good for any amount, with a constant set-up cost δ per delivery and that he always replenishes his stock to the level of Q as soon as the stock is depleted.[7/] A storage cost (rate) d proportional to the level of inventory is charged each instant of time. Denoting by c the unit cost he pays to the factory, we have

$$\begin{aligned}\pi = {} & (p - c) \times \text{average quantity sold 'per week'} \\ & - \delta \times \text{average number of reorders 'per week'} - d \times \text{'weekly'} \\ & \text{average stock level.}\end{aligned} \tag{2'}$$

As indicated in Section 1, we are interested in the decision problem of the middleman. His decision problem is made more difficult because of the random amount of purchases made by individual customers. We now formulate his problem as a sequential statistical decision problem of when to conclude that $p-\hat{p} \neq 0$ or the value of θ has changed, i.e., when to decide that his price is out of line with the price prevailing in the market and when to decide that a changing trend in the amounts sold is due to change in the demand for the goods due to shift in preference and so on.

As a first approximation to this problem, suppose that exogenous change in the demand (shift in the preference) entails that every customer's purchase is changed by the same but unknown amount, θ%, while $p-\hat{p} \neq 0$ changes α and μ in such a way that customers who normally have large demand

are affected more than customers with normally small demand.[8/] This shall be made precise shortly. Suppose that when there is an exogenous shift in the demand schedule then each customer buys $y = (1 + \theta)x$ instead of x. $\theta = 0$ corresponds to the situation before the shift. The amount of purchase now is governed by the probability density function

$$g(y) = \nu \frac{(\nu y)^{\alpha-1} e^{-\nu y}}{\Gamma(\alpha)}$$

where

$$\nu = \frac{\mu}{1+\theta} .$$

Namely, the scale parameter μ is changed to $\mu/1+\theta$, while the shape parameter α remains the same. In other words, the general shift in demand can be detected as a shift in the mean with no change in coefficient of variation defined as

$$C^2(x) = \frac{Var(x)}{[E(x)]^2} ,$$

since the mean changes by $(1+\theta)$ from $\frac{\alpha}{\mu}$ to $\frac{\alpha}{\mu}(1+\theta)$ but the standard deviation also increases by the same proportion from $\sqrt{\alpha}/\mu$ to $\sqrt{\alpha}[(1+\theta)/\mu]$.

Suppose now the price the middleman charges is low compared with the price prevailing elsewhere in the market, $p < \hat{p}$. Then those who have larger demands (and those whose budget allows larger purchases) will buy larger amounts to stock up. If $p > \hat{p}$, however, those who would be normally buyers of large amounts sharply curtail their purchases more than those

with smaller demands.

Keeping our example of cans of soups in mind, we model the type of purchasing pattern which is sensitive to price differentials by positing:

$$q(p) \simeq q(\hat{p}) + \varepsilon(p-\hat{p})q(\hat{p})^{\zeta}, \tag{3}$$

where

$$\varepsilon(p-\hat{p}) \quad \begin{cases} > 0, & p-\hat{p} < 0 \\ = 0, & p-\hat{p} = 0 \\ < 0, & p-\hat{p} > 0 \end{cases}, \tag{3'}$$

and where $|\varepsilon(p-\hat{p})|$ is small for small $|p-\hat{p}|$.

Such a form may be justified by considering a customer whose utility function is given by

$$U = (q_1^{\gamma} + q_2^{\gamma})q_3^{\beta} \qquad 0<\gamma,\beta<1,$$

where good 1 and good 2 are close substitutes. The total number of goods need not be three. This is chosen solely for ease of illustration.

By maximizing U subject to his budget constraint, $I=p_1q_1+p_2q_2+p_3q_3$, the demand for good 1 is given by

$$q_1=\text{const.}p_1^{-1}\left\{1 + \left(\frac{p_2}{p_1}\right)^{\frac{\gamma}{\gamma-1}}\right\}^{-1}$$

with a similar expression for q_2. In other words, if the price difference p_2-p_1 between the substitutes is small, then

$$q_2 \simeq q_1 \left\{1+(p_2-p_1)/(\gamma-1)p_1\right\}$$
$$=q_1+\varepsilon(p_2-p_1)q_1^2 + O(\varepsilon)$$

where

$$\varepsilon(p_2-p_1) \simeq -2(\gamma+\beta)(p_2-p_1)/\ (1-\gamma)\gamma I\ .$$

This is a special case of (3) with $\zeta = 2$.

Eq.(3) leads to the expression of the mean and the variance of purchase,

$$m(p)=m(\hat{p}) + \varepsilon(p-\hat{p})L \tag{4A}$$

where

$$L=E(q^{\zeta})\quad \Gamma(\alpha+\zeta)/\Gamma(\alpha)\mu\zeta > 0,$$

and

$$V(p)=V(\hat{p}) + 2\varepsilon L\zeta/\mu. \tag{4B}$$

When $\zeta = 2$, $L=\alpha(\alpha+1)/\mu^2$.

From these, combining the effects of exogenous shifts, discussed earlier, we postulate that the parameters of the distribution approximately vary with p as

$$\alpha(p) \simeq \alpha(\hat{p})(1+\varepsilon(p-\hat{p}))^2(1+2\zeta\varepsilon(p-\hat{p}))^{-1} \tag{4C}$$
$$\mu(p) \simeq \mu(\hat{p})(1+\varepsilon(p-\hat{p}))(1+2\zeta\varepsilon(p-\hat{p}))^{-1}(1+\theta)^{-1}$$

for $|p-\hat{p}|$ small,

or equivalently

$$m(p) \simeq m(\hat{p})(1+\varepsilon(p-\hat{p}))(1+\theta)$$

and (4D)

$$V(p) = V(\hat{p})(1+2\varepsilon(p-\hat{p}))(1+\theta)^2$$

where we rename, for simplicity,

$$\varepsilon(p-\hat{p})L/m(\hat{p})$$

as $\varepsilon(p-\hat{p})$. Since L and $m(\hat{p})$ are both positive the properties (3') still hold.

Eq(4C)(or (4D)) is basic to our model of the middleman's decision problem.

It follows from (4C or D), then, that the middleman can monitor three sets of parameter combinations in order to determine whether or not a shift in the demand schedule has occurred or his price is out of line with the "prevailing" price, i.e., there are three mutually exclusive cases he can distinguish:

Case 1. If the mean, α/μ, of the demand distribution changes and α does not, then θ (the demand shift parameter) has changed.

Case 2. If α or the coefficient of variation changes, then the asking price is not equal to the market price.

Case 3. If both the mean and the coefficient of variation change, then θ has changed and the price is not equal to market price.

The problem is that of detection or statistical hypothesis testing. A number of standard techniques have been developed and are available in statistical and/or communicating engineering literature. We briefly return to this topic, after we resolve the optimal pricing problem for the middleman.

3. Optimal prices

The middleman sets his price to maximize π of (2'). We show in Appendix 1 that under the assumption that Q is much greater than the average purchase α/μ, we have

$$\pi \simeq (p-c)A - \delta A/Q - d(\frac{Q}{2} + S) + o(^1/_Q) \tag{5}$$

where

$A \simeq \rho m$ is the average quantity sold 'per week', ρ is the (average) customer arrival rate, $m = \alpha/\mu$

A/Q is the average number of reorders 'per week'

and

$\frac{Q}{2} + S$ is the average stock level. In Appendix 1, it is shown that

$$S = \frac{m}{4} - \frac{k}{4\mu} \tag{6}$$

where $k = O(1)$.

In the above, $dQ/2$ is the cost of storage 'per week' if the quantity Q is sold at a constant rate. So long as Q is fixed, it is a fixed cost independent of prices. There is a correction term, dS, to the average cost of storage due to the fact that customers do not buy same quantities and only a finite number of them arrive per period. S is not zero even if the same number of customers are assumed to arrive at the store at a uniform rate, so long as they do not buy the same quantity. Thus, $S \neq 0$ reflects in an essential way the random purchase assumption of this model.

From (5), the first order condition for the maximization of the profit rate is

$$p^* - c - \delta/Q = -\frac{A}{A'} + d\,\frac{S'}{A'} \tag{7}$$

where ' denotes the derivative with respect to p, p* is the optimal price [9/] and where

$$A'/A = \rho'/\rho + \frac{m'}{m}. \tag{8}$$

Let

$$e = -\partial \ln A/\partial \ln p,$$

and

$$f = -\frac{1}{e}\partial \ln s/\partial \ln p$$: the price induced ratio of % change in s over % change in A.

Substituting these into (6), we obtain the expression for p*

$$p^* = \{c + \delta/Q + dfS/A\}/(1-1/e).$$

In the above expression for the optimal price, ds/A is the ratio of the storage cost deviation (from the weekly average storage cost) over the average weekly sales. Higher this ratio, higher p*. When everybody purchases a same quantity, then it disappears completely. The expression (1-1/e) shows that higher e, lower p*.

The maximum average profit is

$$\pi^* = -dQ/2 + \{(c + \delta/Q)A/(e-1)\} + ds(\frac{f\,e}{e-1} - 1).$$

An exogenous shift θ changes the last two terms of the above, i.e., the part of the profit dependent on p changes to

$$\{(c + \delta/Q)A/(e-1) + ds(\frac{fe}{e-1} - 1)\}\ (1 + \theta).$$

Thus, if the middleman has chosen Q at which π* is zero, then θ > 0 now makes his average profit positive and θ < 0 would make it negative.

In evaluating e and f, we need price eleaticities of α, μ, s and ρ. At the end of Section 2 we posited (4). From these relations we can obtain

the needed price elasticities of them.

We assume $\rho' = 0$. In other words, the knowledge that the middleman's price is different from the price prevailing in the market does not spread instantaneously hence does not affect in the short run the rate at which the customers come to him.[10] See Section 5 for the discussion of $\rho' \neq 0$. Combining the expressions for m'/m and μ'/μ from (4D) into (6) and (7), we obtain the expression for the middleman's price when he is in line with $\hat{p}$ as

$$p = \hat{p} = c + \delta/Q - \frac{1}{\varepsilon'(0)} + \frac{d}{4\rho}\left\{1 - \frac{k(2\zeta-1)}{\alpha(\hat{p})}\right\}, \tag{8}$$

since

$$A'/A \simeq \frac{\varepsilon'(p-\hat{p})}{1+\varepsilon(p-\hat{p})} = \frac{\varepsilon'(0)}{1+\varepsilon'(0)(p-\hat{p})}, \tag{9}$$

and

$$\frac{S'}{A'} = \frac{1}{4\rho}\left\{1 - \frac{k(2\zeta - 1)}{\bar{\alpha}} + \frac{2k(2\zeta - 1)}{\bar{\alpha}}\varepsilon(p - \hat{p})\right\} \tag{10}$$

for $|p-\hat{p}|$ small, where $\bar{\alpha} = \alpha(\hat{p})$.

Change in $\hat{p}$

The right hand side of (7) together with (9) and (10) enables the middleman to follow $\hat{p}$ when it changes under mild conditions on the assumed manner by which the distribution of q is sensitive to price differentials. We state this as

Proposition

Suppose $\hat{p}$ shifts to a new value and stays at the new value. Then the middleman following the price equation (7) can eventually restore the price differential to zero, if

$$\left|\frac{d(2\zeta-1)k\varepsilon'(0)}{2\rho \quad \alpha(\hat{p})}\right| < 1 \ . \tag{11}$$

See proof in Appendix 2.

The condition of convergence (11), rewritten as

$$|k|\,|\varepsilon'(0)| < \frac{2\rho\alpha(\hat{p})}{d(2\zeta - 1)} \ ,$$

shows that the price sensitive behavior of the quantity purchased can't be too great, otherwise the middleman tends to overcorrect. It also shows that higher the storage cost more unstable the price correction behavior is likely to be for the same $|\varepsilon'(0)|$. In other words, the higher d, the smaller $|\varepsilon'(0)|$ must be for this price adjustment method to be stable. The more frequently customers arrive, the higher $\varepsilon'(0)$ could be.

In the special case of $\zeta = 2$, (11) becomes (recalling that $L\varepsilon/m$ is renamed as ε in (4))

$$\left|k\varepsilon'(0)\right| < \frac{2\rho}{3d}\,\frac{\alpha\mu}{(\alpha + 1)} < \frac{2\rho}{3d}\,\frac{m(\hat{p})}{V(\hat{p})} \tag{12}$$

showing that a smaller value of the ratio of m/V requires a smaller value of $|\varepsilon'(0)|$ for stable adjustments. In other words, as variability of individual's purchase become larger, less sensitive the quantity

purchased must be to price differential for the middleman to adjust his prices successfully.

Change in θ

When the price difference is zero, the middleman's price is given by (8). In (8), none of the terms will change when general shifts in demand occur since we assume that ρ is not affected by θ in this model in the short run.

Thus we have

Proposition

The optimal price of the middleman is the same for $\theta \neq 0$ in the short run. This may be seen also directly from the expression for the long run profit (5). In (5), the portion of π that is price sensitive (i.e., all terms except $-dQ/2$) is affected equally by $(1 + \theta)$.

4. Parameter Estimates and Their Revisions

Obviously the price that the middleman sets is strongly dependent on his beliefs about the customer arrival rate, the parameters of the demand schedule and their elasticities. Because the parameters are not known perfectly and are subject to change without his knowledge, the middleman must estimate them and be prepared to revise his beliefs in the face of contradiction with observed data.

The statistical problem he faces is, first of all, that of hypothesis testing and secondly that of parameter estimation: he must decide to keep his current beliefs (point estimates) of the parameters or decide to reject them and revise his beliefs (substitute new point estimates).

Since he can resort only to statistical processing of the sales records he possesses to arrive at whatever conclusions such as he must revise his estimates of the parameters or the sales data he has does not refute his currently held estimates and so on, he will need sales records involving large number of customers to reduce the probabilities of wrong decisions.[11/] The higher the rate of customer arrivals, the better he is able to cope with shorter run phenomena. However, it would place an unreasonable computational burden on the middleman to require him to update his parameter estimates after each customer. Even if he did update them that often it is highly unlikely that he would want to continually vary his prices based on the considerations of information cost and of the reliability of his decisions. We thus assume that all the observed data is pooled from the time of his most recent price change.

The hypothesis is that his current beliefs are correct. The alternate hypothesis is that they are wrong. Such a test is called composite hypothesis testing in the statistical literature. It is known that the likelihood ratio test has optimal asymptotic properties under some regularity conditions which are also sufficient for the existence of asymptotically normally distributed maximum likelihood estimates, and in many cases has optimal properties for finite samples as well. Further it is known that the logarithm of the likelihood ratio asymptotically has the chi-square distribution and the maximum likelihood estimates asymptotically coincide with the maximum chi-square estimates, see Cramer, Wilkes. Since the maximum likelihood estimates for the parameters of Gamma distributions are asymptotically normally distributed and asymptotically efficient, a simple

test that the middleman can perform to determine whether or not his beliefs are correct is to use chi-square estimates of the parameters and use the table to accept or reject the hypothesis, or more simply to monitor the maximum likelihood estimate of each parameter and to reject his belief in any parameter estimate which escapes from a band of width 3 three standard deviations (given in the previous asymptotic normal distributions) on each side of his current belief in the parameter value. If a parameter value is rejected then the current maximum likelihood estimate of that parameter is assumed to be the correct value. This new parameter value is then tested in the manner just described.

See Appendix for the expression of the maximum likelihood estimates.

5. Conclusion and Discussion

The paper shows the importance of the inventory level in a model of a middleman in which random quantity purchase by individual customers is explicitly modelled. In other words quantity purchased varies from customers to customers. The probability distribution of quantity purchased is then assumed to change with price.

The pricing behavior of the middleman remains the same when the customers are assumed to arrive as a Poisson process. This complicates the computation of the average inventory level somewhat. Its expression is the same up to $0(1/Q)$.

The condition (11) or (12) shows that the middleman need not know the parameters α, μ and ρ exactly so long as they satisfy the indicated inequality for him to be able to "track" $\hat{p}$ eventually.

For simplifying the exposition we assume $\rho' = 0$ in this paper. As indicated in the main body of the paper this does not change the formula for p^* since ρ' enters into p^* only through A'/A where A is the average sales 'per week'. It is a simple matter to include a test for ρ = constant. To consider the price-induced change in ρ, however, it is more realistic to model the customer arrival as a stochastic process such as a Poisson process and incorporate another hypothesis test for $\dot{\rho}$ = constant. It is possible to give more structure to price-sensitive behavior of ρ by modelling the gradual spread of price information explicitly. See for example Phelps and Winter in Phelps, for one such model.

How might he know that theories or beliefs he holds above the economy are being systematically refuted by messages or signals they receive? This paper formulates this question as a sequential hypothesis testing problem and illustrates the formulation in some detail for a middleman with inventory of nonperishable consumption goods. Even though the details of discussions may be specific to this simple example, the outline of analysis indicated in this paper is believed to be useful in a broader context. For example, the idea of using price dependent parameters to model the collective behavior of consumers may find applications in other areas.

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Footnotes

1. The role of buffer stocks in Keynesian models has been discussed by Leijonhufvud, [5].

2. This example has been suggested by A. Leijonhufvud.

3. Instead of assuming that gathering and processing information related to $\hat{p}$ is costly and/or time consuming, we may assume that the middleman is a new entrant to the market or has just introduced a new line of goods or opened a new store. The author owes this interpretation to M. Nabli. We do not however discuss the aspect of entry to the market, assuming the market is closed to new entry.

4. An alternate possibility is to assume that a customer has the probability a of purchasing a unit of the good and the probability 1-a of not purchasing the good. Such a model may be appropriate when the good in question is a durable or capital good. The kinds of goods that fit our model would be more in the nature of nonperishable consumer goods, cans of soups for example.

5. Customer arrivals can be modeled as a stochastic point process, for example, a Poisson process.

6. The explicit adoption of Γ-distributions as the demand distribution of an individual customer is not cruical for the purpose of analysis in this paper. Its use permits us to derive explicit expressions for such items as the average cost of storage, etc. as functions of the location and scale parameters of the demand curve which facilitates some comparative static analysis. Besides, the use of the Γ-distribution is not very restrictive, appropriate choice of the location and scale parameters permit a wide range of possible shapes of the demand distribution to be approximated.

7. He follows therefore (s,S) policy where s=0 S=Q. Here s may be taken to be zero due to the instantaneous delivery assumption. Given fluctuating demands, he will attribute the demand variation in part to his price being not in line with the average market price and in part attribute the variation to the exogenous change in the economic conditions. How this is done is outlined later in the paper. He would generally change both Q and the price p. In this paper, we focus attnetion to the pricing scheme while holding Q fixed.

8. This has been suggested by A. Leijonhufvud.

9. Note that the price induced change in the customer arrival rates appear in p^* only through A'/A, i.e., as the changes in the average sales 'per week'. It is therefore of no great loss of generality to assume $\rho' = 0$ in the short run and assume A'/A to be entirely due to m'/m.

10. See footnote 9.

11. There are two types of errors; concluding erroneously that revision is called for and the error of not detecting changes in the parameter values. When the sample is small, he will not be able to make decisions with any reliability. In other words, he will not be able to catch very transient or temporary fluctuations in θ, for example.

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Appendix I

Derivation of the Long Run Profit Function

To obtain the long run average expression of π, as defined by (2'), we need compute the three terms in (2'). They are stated as Lemma 1, 2 and 3. We assume that customers arrive at the middleman's store at a uniform rate of ρ customers 'per week'.

Let T_Q be the random variable denoting the time interval (weeks) between reorder and let N_Q be the number of customers (random variable) between reorder.

By definition

$$\rho E(T_Q) = E(N_Q).$$

Lemma 1

$$E(N_Q) = \frac{Q}{m} + \frac{\alpha + 1}{2\alpha} + o\left(\frac{1}{Q}\right).$$

Note that

$$\frac{\alpha + 1}{2\alpha} = \frac{m^2 + V}{2m^2}$$

where

$$m = E(q), \text{ and } \qquad V = \mathrm{Var}\ (q)\ .$$

Proof

Let q_i be the quantity purchases by the i-th customer.

Then, denoting probability events by [ ], we have

$$[N_Q = r] \iff [\textstyle\sum_{i=1}^{r-1} q_i < Q \leq \sum_{i=1}^{r} q_i] ,$$

where

$$[\sum_{i+1}^{r-1} q_i < Q \leq \sum_{i=1}^{r} q_i] = [\sum_{i=1}^{r-1} q_i < Q, \text{ and } Q \leq \sum_{i=1}^{r} q_i]$$

and where

$$[\sum_{i=1}^{r-1} q_i < Q] = [\sum_{i=1}^{r-1} q_i < Q \text{ and } \sum_{i=1}^{r} q_i < Q] \cup [\sum_{i=1}^{r-1} q_i < Q \text{ and }$$

$$\sum_{i=1}^{r} q_i \geq Q].$$

These two events on the right-hand side are mutually exclusive, hence

$$P[\sum_{i=1}^{r-1} q_i < Q \leq \sum_{i=1}^{r} q_i] = K_{r-1}(Q) - K_r(Q) \tag{A.1}$$

where

$$K_r(Q) = P[\sum_{i=1}^{r} q_i < Q], \quad r = 1, 2, \ldots \tag{A.2}$$

$$K_0(Q) = 1 \quad , \quad Q \geq 0 .$$

We have

$$\begin{aligned} E(N_Q) &= \sum_{r=0}^{\infty} r \, P[N_Q = r] \\ &= \sum_{r=0}^{\infty} r\left(K_{r-1}(Q) - K_r(Q)\right) \\ &= \sum_{r=0}^{\infty} K_r(Q). \end{aligned} \tag{A.3}$$

We assume the series is absolutely convergent.

We can compute $K_r(Q)$ defined in (A.2) by the Laplace transform.

From the assumption, q_i are independently and identically distributed with the probability density function $g(q)$. Denote its Laplace transform by $\hat{g}(s)$. Then the Laplace transform of the density $\sum_{i=1}^{r} q_i$ is $[\hat{g}(s)]^r$.

By definition, $\hat{g}(s)$ is given b··

$$\hat{g}(s;\theta) = \int_0^\infty \frac{e^{-sq}\ \mu^\alpha q^{\alpha-1} e^{-\mu q}}{\Gamma(\alpha)}\, dq$$

$$= \mu/(\mu + s)^\alpha, \quad s \neq \mu.$$

Denote the Laplace transform of $K_r(Q)$ by $K_r(s)$. We have

$$K_r(s) = \frac{1}{s}\left(\frac{\mu}{\mu + s}\right)^{\alpha r}. \tag{A.4}$$

From (A.3) and (A.4), the Laplace transform of $E(N_Q)$, $L[E(N_Q)]$ is given by

$$L[E(N_Q)] = \frac{1}{s\left[1 - \left(1 + \frac{s}{\mu}\right)^{-\alpha}\right]}$$

We assume α to be a positive integer so that this is a rational function.

An asympotic expression of the above for large Q can be obtained by expanding the above as the Laurent series expansion in s,

$$LE(N_Q) = \frac{\mu}{\alpha}\left\{\frac{1}{s2} + \frac{\alpha + 1}{2\mu}\ \frac{1}{s} + 0(1)\right\}.$$

We can obtain $E(N_Q)$ from the above by taking its inverse Laplace transform

$$E(N_Q) = \frac{\mu}{\alpha}\left\{Q + \frac{\alpha + 1}{2\mu} + 0(1)\right\}. \tag{A.5}$$

Using the similar technique, we can establish

Fact

$$\mathrm{Var}(T_Q) < \infty.$$

Lemma 2

The average number of reorder per week is $1/E(T_Q)$.

Proof

Consider a large number M of reorders. It spans the period of $T_1 + T_2 + \ldots + T_M$ where T_i is the time interval of i-th reorder. Each T_i is independently and identically distributed with T_Q discussed above. Thus by the Kolmogorov's strong law of large numbers.

$$\frac{M}{\Sigma_{i=1}^{M} T_i} \longrightarrow \frac{1}{E(T_Q)} \qquad \text{a.s.}$$

Lemma 3

The average stock level, S, is given by

$$E(S) = Q \simeq \frac{m}{2}\left(E(N_Q) - 1\right) \tag{A.6}$$

$$= \frac{Q}{2} + \frac{\alpha-1}{4\mu} + 0\left(\frac{1}{Q}\right)$$

Proof

We establish

$$E(S|N_Q = r) \simeq Q - \frac{m}{2}(r-1)$$

Then (A.6) follows immediately.

The average stock level is by definition

$$S = (\text{time integral of the stock}) / T_Q.$$

When r customers arrived in one cycle, then

$$S = \left\{Q + (Q - \delta_1) + \ldots + (Q - \Sigma_1^{r-1} q_i)\right\}/r$$

$$E(S|N_Q = r) \simeq Q - m\,(1 + 2 + \ldots + r-1)/r$$

$$= Q - \frac{m}{2}(r-1),$$

where the approximation consists in replacing

$$E(q_i|\ \Sigma_{j=1}^{r-1} q_j < Q) \text{ by } E(q_i) = m.$$

We can check goodness of this approximation by computing E(S) in an alternate way.

Consider a large number N of re-order cycles. Let T_i and S_i be the duration and the time integral of the inventory of the i-th cycle. The average inventory, then, is given by the limit of I_N, where

$$I_N = \sum_{i=1}^N S_i / \sum_{i=1}^N T_i \tag{A.7}$$

where T_i are independently and identically distributed. The random variable S_i is also i.i.d.

Let

$$\overline{S} = ES_i$$

$$\overline{T} = ET_i \quad , i = 1, \ldots, n.$$

(A.7) is expressible as

$$I_N = \frac{\bar{S} + \sum_{i=1}^{N} (S_i - \bar{S})/N}{\bar{T} + \sum_{i=1}^{N} (T_i - \bar{T})/N}$$

$$= \frac{\bar{S}}{\bar{T}} \frac{1 + \frac{1}{\bar{S}N} \sum_{i=1}^{N} (S_i - \bar{S})}{1 + \frac{1}{\bar{T}N} \sum_{i=1}^{N} (T_i - \bar{T})} .$$

By the Kologorov's strong law of large numbers, we have

$$\frac{1}{N} \sum_{i=1}^{N} (S_i - \bar{S}) \to 0 \qquad \text{a.s.} \qquad \text{as } N \to \infty,$$

and

$$\frac{1}{N} \sum_{i=1}^{N} (T_i - \bar{T}) \to 0 \qquad \text{a.s.} \qquad \text{as } N \to \infty.$$

Thus

$$I_N \to \bar{S}/\bar{T} \qquad \text{a.s.}$$

From Lemma 1, we know that

$$\bar{T} = \frac{\mu}{\rho\alpha}[Q + \frac{\alpha+1}{2\mu} + o(1)].$$

We next compute $\bar{S}$.

Suppose $N_Q = r$. Then the time integral of the inventory is

$$S = \frac{1}{\rho} [Q + (Q-q_1) + \ldots + (Q-\Sigma_1^{r-1}\delta_i)]$$

Assume $Q >> 1$ so that

$$E(q_1 | q_1 < Q) \simeq E(q_1) + o(1)$$
$$= \alpha/\mu + o(1),$$
$$E(q_1 + q_2 | q_1 + q_2 < Q) \simeq \frac{2\alpha}{\mu} + o(1/Q) \quad \text{etc.}$$

Then

$$E(S | N_Q = r) \simeq \frac{1}{\rho}[rQ - \frac{\alpha}{\mu}(1 + 2 + \ldots + r-1)] + o(1)$$
$$= \frac{1}{\rho}[rQ - \frac{\alpha}{\mu}\frac{r(r-1)}{2}] + o(1).$$

Thus

$$\overline{S} = E(S) \simeq \frac{1}{\rho}[\overline{r}\ Q\ \frac{\alpha}{\mu} - \frac{\overline{r(r-1)}}{2}\ ] + o(1)$$

We can compute $\overline{r^2} = \Sigma_{r=0}^{\infty} r^2 \left(K_{r-1}(Q) - K_r(Q)\right)$

analogously to our computation of $\overline{r}$ in Lemma 1,

$$\overline{r(r-1)} = \left(\frac{\mu}{\alpha}\right)^2 Q^2 + \frac{2\mu}{\alpha^2} Q + \frac{1-\alpha^2}{6\alpha^2} + o(1),$$

and

$$\overline{r} = \left(\frac{\mu}{\alpha}\right) Q + \frac{\alpha+1}{2\alpha} + o(1).$$

Substituting these, we obtain

$$\rho\overline{S} = \frac{\mu}{2\alpha} Q^2 + \frac{\alpha-1}{2\alpha} Q - \frac{1-\alpha^2}{12\alpha\mu} + o(1),$$

or

$$E(S) \simeq \frac{Q}{2} + \frac{\alpha-3}{4\mu} + \frac{(\alpha+1)(\alpha-7)}{24\,\mu^2}\frac{1}{Q} + o(1/Q). \qquad (A.8)$$

Comparing (A.6) with (A.8), we see that these expressions differ by O(1) since they employ different approximations. It is reassuring that they both agree on the leading term, however. We denote the average stock level then, as

Proposition

$$E(S) \simeq \frac{Q}{2}\,[1 + \frac{\alpha-k}{2\mu Q} + o(1/Q)]$$

where $K = O(1)$.

In other words, the average stock level is approximately Q/2 as expected since the stock level fluctuate between 0 and Q and a correction factor depending on the average amount of purchase per customer α/μ and on Q.

Appendix 2

When the middleman is at equilibrium with the 'prevailing' market price, his price is given by (8) or writing $\overline{\alpha}$ for $\alpha(\hat{p})$,

$$p = \hat{p} = c + \frac{\delta}{Q} + \frac{d}{4\rho}\left(1 - \frac{(2\zeta - 1)k}{\overline{\alpha}}\right) - \frac{1}{\varepsilon'(0)}$$

Suppose the price prevailing elsewhere in the market changes from $\hat{p}$ to $\hat{p}'$.

Take this to have occurred at time 0. Since the reference price has shifted from $\hat{p}$ to $\hat{p}$, the middleman notices this as changes in s'/A' and in m/m'.

At time 1 (in an appropriate time unit the middleman uses to revise his prices such as a week, a month and so forth), the middleman's price becomes from (8), (9) and (10),

$$p_1 = \hat{p} - \lambda(\hat{p} - \hat{p}$$

where

$$\lambda = 1 - \frac{d(2\zeta - 1)\,k\varepsilon}{2\rho\alpha(\hat{p}')} \quad .$$

In general, after t 'weeks' of price adjustment, the middleman notices that his estimate of α and μ are still not correct and adjust his price by

$$p_{t+1} = p_t - \lambda(p_t - \hat{p}')$$

$$p_o = \hat{p} \quad .$$

This adjustment equation converges to

$$\lim_{t\to\infty} p_t = \hat{p}' \quad ,$$

provided $|1 - \lambda| < 1$ or

$$\frac{d(2\zeta - 1)k|\varepsilon'(0)|}{2\rho\,\alpha(\hat{p}')} < 1.$$

APPENDIX 3

Maximum Likelihood Estimates and Asymptotic Distributions

We summarize some useful facts on the maximum likelihood estimates for easy reference.

Assume the estimation is to be based on the observations of n customers. The demand density is assumed to have a Gamma distribution.

The likelihood function is

$$L(q,\mu,\alpha) = \mu^{n\alpha} \prod_{i=1}^{n} q_i^{\alpha-1} e^{-\mu \sum_{i=1}^{n} q_i} / \Gamma^n(\alpha) .$$

Taking the ln of it and maximizing it with respect to the desired parameters yields the maximum likelihood estimates,

$$\hat{\mu} = \hat{\alpha} / \frac{1}{n} \sum_{i=1}^{n} q_i ,$$

where

$\hat{\alpha}$ is the solution to

$$\log \hat{\alpha} - \psi(\hat{\alpha}) - \ell n \frac{1}{n} \sum q_i + \frac{1}{n} \sum \ell n\, q_i = 0 ,$$

and where we define

$$\psi(\hat{\alpha}) = \frac{d}{d\alpha} \ln \Gamma(\hat{\alpha}) = \frac{\Gamma'(\hat{\alpha})}{\Gamma(\hat{\alpha})} .$$

Asymptotically $\quad \psi(\hat{\alpha}) \approx \ln \hat{\alpha} - \frac{1}{2\hat{\alpha}} - \frac{1}{12\hat{\alpha}^2} + o(\frac{1}{\hat{\alpha}^2})$.

If $\hat{\alpha}$ is large enough then $\hat{\alpha}$ is the solution to

$$\frac{1}{2\hat{\alpha}} - \ln \frac{\sum q_i}{n} + \frac{1}{n} \sum \ln q_i = 0$$

or

$$\hat{\alpha} = \frac{1}{2n[\ln \sum_{i=1}^{n} qi - \frac{1}{n} \ln \sum_{i=1}^{n} q_i]} .$$

Cox and Lewis [p137,3] show that the sample coefficient of variation is an asymptotically efficient estimator of α as α becomes large. It is a simplier and consequently perhaps more desirable estimate of α.

$$\hat{\alpha}^* = \sqrt{\frac{(\frac{1}{n} \sum q_i)^2}{\frac{1}{n} \sum q_i^2 - (\frac{1}{n} \sum q_i)^2}}$$

Continuing with the other maximum likelihood estimates

$$\hat{m} = \frac{1}{n} \sum_{i=1}^{n} q_i$$

$$\hat{V} = \frac{\frac{1}{n} \sum_{i=1}^{n} q_i}{\hat{\alpha}^{\frac{1}{2}}}$$

Finding the expected value of $\text{-}1$ of the second derivate of the log likelihood function leads to the expression for the variance of the asymptotic normal distribution of the parameter estimates (Wilks, Cramer),

$$\hat{\alpha} \sim N\left(\alpha, \frac{1}{n[\psi'(\alpha)-\frac{1}{\alpha}]}\right)$$

and

$$\hat{\mu} \sim N\left(\mu, \frac{\mu^2}{n[\alpha-\frac{1}{\psi'(\alpha)}]}\right)$$

where $\psi'(\alpha) = \frac{d}{d\alpha}\psi(\alpha) \approx \frac{1}{\alpha} + \frac{1}{2\alpha^2} + \frac{1}{6\alpha^3} + o(\frac{1}{\alpha^3})$.

Note that

$$\text{Var }(\hat{\mu}) \simeq \frac{2\mu^2}{n} + 0\left(\frac{1}{\alpha n}\right).$$

We see that $\text{Var}(\hat{\mu})$ is independent of α up to $o(\frac{1}{\alpha n})$. Thus for αn of the order 10^4, then $\text{Var}(\hat{\mu})^{\frac{1}{2}}$ is of the order 10^{-2}. Thus for all practical purposes, the width of the estimation confidence interval may be taken to be independent of α.